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Quantization of Nonintegrable Maps

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Using Heisenberg's matrix formulation of quantum mechanics, a method is given for quantizing volume-preserving polynomial mappings. The energy levels of the linear map are obtained exactly and those of the cubic, nonintegrable map are obtained approximately and numerically.

KEY WORDS: Nonintegrable maps; quasi-energy levels; matrix mechanics.

1. INTRODUCTION

Dear Nico,

May you continue to exemplify your "Physics is a Way of Living, not just a way to make a living"[1] for many more student generations to come. Watching you pace back and forth ("polar-bearing," to coin a Hollandicism) while casting before us students the pearls of physics mixed with wisecracks, paradoxes, and one-liners, we learned and saw that it is a fascinating way of living. Watching your Triumph Spitfire tear away, we surmised there might even be a way to make a living along that dangerous road. We admired, and admire, your insights and extremely clear opinions. The only time I remember uncertainty creeping in was when a local paper asked your choice for some position in the University: "I don't know whether I prefer Mickey Mouse or Donald Duck." We, on the other hand, were certain that either one would have been preferable.

One of the many nonlinear problems of interest to van Kampen[1-7] is "quantum chaos."[5-7] At this time it is not clear whether there exists any chaos in quantum systems.[4] Here I consider the related problem of trying to quantize a *typical* nonintegrable system, i.e., a Hamiltonian system which, therefore, is neither completely "ergodic" (/chaotic) nor completely

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"integrable."[8,10] We do know how to quantize two classes of *non*typical Hamiltonian systems, i.e., integrable systems and a few "ergodic" non-integrable ones.[5–8,16] It is an open problem how to quantize typical non-integrable systems, notwithstanding the fact that this covers "almost all" Hamiltonians.[8,10]

Classically we usually find it simpler to study the corresponding non-integrable *mappings* (:difference equations) rather than the original differential equations.[8,10–12] Here, I propose to quantize some typical, "elliptic" mappings of the plane and find their energy levels, employing Heisenberg, Born, and Jordan's original *matrix mechanics* formulation of quantum mechanics.[13–15] Some maps (billiards) have been treated earlier, semiclassically or numerically, by various authors, employing the Wigner and/or familiar Schrödinger formalisms.[5–7,16,27–32]

In Section 2, Heisenberg's matrix mechanics, in a form applicable to mappings, is introduced. It contains most of the notation and formalism we shall need. Sections 3 and 4 are more applied. In Section 3, the simplest possible example is provided, by quantizing the linear "harmonic" ("elliptic") map exactly. I compare it with the familiar quantization of the full (4-dimensional) Hamiltonian, which reproduces this linear map as its "surface of section map." Apart from a trivial change of time (/energy) scale, we exactly recover the levels due to the motion "parallel" to the plane of the map. The levels due to the motion "perpendicular" to the plane of the map are not obtained, of course, but are not excluded by the present approach. In Section 4.1, I calculate the "levels" of a cubic map, approximately and numerically. The difference between any two subsequent levels (divided by $\hbar$) is smaller than, but stays close to, the rotation numbers for the "corresponding" classical orbits of the map. These level seperations for the cubic map are smaller than those for the linear map, but not much smaller, in the energy range considered here. In Section 4.2 I study the time evolution of a mixed state. Since I do not employ physical units in this paper, Planck's constant $(/2\pi):\hbar$ merely is a (positive) number here whose value we are at liberty to choose. The maximum values of $\hbar$ for which our results stay bounded are larger in Section 4.2 than in Section 4.1.

2. QUANTUM MATRIX MAPS

A form of matrix mechanics is developed here which is applicable to mappings. No previous knowledge of matrix mechanics is assumed. I use systems with two degrees of freedom as an example of the procedure and notation.

Given a real-analytic Hamiltonian system with Hamiltonian H (in 4-dimensional phase space), we consider, for just *one* pair of its conjugate

variables, the relation between their values at integer moments in time, i.e., its "stroboscopic" mapping T,[10-12]

$$T: \quad q_{t+1} = g(q_t, p_t), \quad p_{t+1} = f(q_t, p_t), \quad t = ..., -1, 0, 1, 2,... \tag{2.1}$$

with g and f real analytic functions. This is a mapping of the q, p plane to itself, also known as a "time-1" map or a (curved) "surface of section" map (Poincaré).[10-12] The unit of time in Eq. (2.1) is chosen to be the time of return of an orbit to a (curved) "surface of section" ("close to" a flat surface[12]. When the map (2.1) is generated from a "larger" (here 4-dimensional) Hamiltonian system, the two conjugate variables q and p can usually be chosen such that the Jacobian (determinant) of the map (2.1) equals 1 everywhere in the q, p plane (cf. Poincaré invariants).[9-12] Hence this "area-preserving" property can be reexpressed as

$$\{q_{t+1}, p_{t+1}\} = \{g, f\} \quad (\equiv \partial g/\partial q_t \, \partial f/\partial p_t - \partial f/\partial q_t \, \partial g/\partial p_t) = 1 \tag{2.2}$$

at all values of q, p, employing the classical Poisson brackets $\{\cdots\}$. So we see that $\{q_{t+1}, p_{t+1}\} = \{q_t, p_t\}$, a "*mapping* constant of the motion." For planar maps this has a similar function, and similar consequences, as Liouville's theorem has in the full (4-dimensional) phase space.

In the original Heisenberg matrix approach to quantum mechanics[13-15] scalar variables are replaced with *infinite-dimensional matrices*, e.g., the q, p become complex infinite matrices $\mathbf{Q}$, $\mathbf{P}$ which are Hermitian, i.e., $\mathbf{Q}^\dagger = \mathbf{Q}$, $\mathbf{P}^\dagger = \mathbf{P}$, with real eigenvalues.

The most unusual aspect of matrix mechanics to people solely familiar with the later Schrödinger approach is that the *(time-dependent)* $\mathbf{Q}, \mathbf{P}$ matrices do satisfy the usual *classical*, e.g., *nonlinear*, equations of motion, since Heisenberg and others wanted to maintain the classical equations of motion in their original approach[13-15] [cf. Eqs. (2.12)]. Here the mapping T in (2.1) must be satisfied and our time-dependent "matrix map" $\mathbf{T}$ becomes

$$\mathbf{T}: \quad \mathbf{Q}_{t+1} = \mathbf{g}(\mathbf{Q}_t, \mathbf{P}_t), \quad \mathbf{P}_{t+1} = \mathbf{f}(\mathbf{Q}_t, \mathbf{P}_t), \quad t = ..., -1, 0, 1, 2,... \tag{2.3}$$

with $\mathbf{g} = \mathbf{g}^\dagger$, $\mathbf{f} = \mathbf{f}^\dagger$. Quantum extensions $\mathbf{g}$, $\mathbf{f}$ of the classical g, f in our map (2.1) are obtained from the Taylor series expansions of $g(q, p)$, $f(q, p)$ and the replacements $q \to \mathbf{Q}$, $p \to \mathbf{P}$. In general, there is more than one such "qm extension" $\mathbf{f}$, each with the same classical analogue (/limit) f, as we see below Eq. (2.4). For the maps I employ in Sections 3 and 4, however, the extensions $\mathbf{g}(\mathbf{Q}, \mathbf{P})$, $\mathbf{f}(\mathbf{Q}, \mathbf{P})$ are *unique*. If no confusion is likely to arise, I often drop the index t.

Essential to quantum mechanics is the commutator[13-15,17]

$$[\mathbf{Q}_t, \mathbf{P}_t] \equiv \mathbf{Q}_t\mathbf{P}_t - \mathbf{P}_t\mathbf{Q}_t = i\hbar\mathbf{I} \tag{2.4}$$

where $\mathbf{I}$ is the identity matrix and $\hbar$ is Planck's constant divided by 2π. This constraint (2.4) on $\mathbf{P}_0$, $\mathbf{Q}_0$, for instance, is of course different from the classical initial conditions, which can *both* be chosen arbitrarily. Formally, it is the main difference between quantum and classical mechanics, cf. Section 4.2. Repeated application of this "qm commutator" shows that many $\mathbf{Q}$, $\mathbf{P}$ permutations in the qm (series) extensions $\mathbf{g}$, $\mathbf{f}$ are in fact the same, or cancel, but not all: for example, the two Hermitian observables $\frac{1}{2}(\mathbf{Q}^3\mathbf{P}^2 + \mathbf{P}^2\mathbf{Q}^3)$ and $\mathbf{QPQPQ}$. While both expressions have the same classical analogue, q^3p^2, they are different operators which *cannot* be equated via the qm commutator (2.4).[18] The maps we employ in Sections 3 and 4, however, do have unique extensions. In general, we prefer those extensions $\mathbf{g}$, $\mathbf{f}$ that satisfy as many of the "classical" equations and conditions as possible, e.g., the "ap property" (2.2).

The Poisson brackets of Eq. (2.2) have a familiar qm extension, since[17]

$$\lim_{\hbar \to 0} (1/i\hbar)[\mathbf{g}, \mathbf{f}] = \{\mathbf{g}, \mathbf{f}\} \tag{2.5a}$$

as could be checked by repeated and tedious application of the qm commutator (2.4) to the series definitions of $\mathbf{g}$, $\mathbf{f}$.[13–15,17,19–21],[2] The derivatives in $\{\cdot\}$ are the usual derivatives here, but with respect to the appropriate variables $\mathbf{Q}$, $\mathbf{P}$, while maintaining the original ordering of the $\mathbf{Q}$, $\mathbf{P}$'s in $\mathbf{g}$, $\mathbf{f}$. For instance,

$$[\mathbf{Q}, \mathbf{PQP}] = i\hbar\, \partial(\mathbf{PQP})/\partial\mathbf{P} = i\hbar(\mathbf{QP} + \mathbf{PQ})$$

Using Eq. (2.4), it is easily checked that whenever $\mathbf{g}$ (or $\mathbf{f}$) is *linear* in $\mathbf{Q}$, $\mathbf{P}$, Eq. (2.5a) is an *identity* for any value of $\hbar$, i.e., taking $\hbar \to 0$ is not necessary:

$$(1/i\hbar)[\mathbf{Q}, \mathbf{f}] = \{\mathbf{Q}, \mathbf{f}\} = \partial\mathbf{f}/\partial\mathbf{P} \tag{2.5b}$$

Hence, instead of the area-preserving property (2.2), we require the "AP property"

$$(1/i\hbar)[\mathbf{g}, \mathbf{f}] = \mathbf{I} \tag{2.6}$$

[2] The precise relation between $[\mathbf{g}, \mathbf{f}]$ and $\{\mathbf{g}, \mathbf{f}\}$ for any finite value of $\hbar$, and not just in the limit $\hbar \to 0$ as in Eq. (2.5a), can be obtained by induction, via repeated and tedious application of the qm commutator (2.4) to the series definitions of the functions $\mathbf{g}(\mathbf{Q}, \mathbf{P})$, $\mathbf{f}(\mathbf{Q}, \mathbf{P})$. McCoy[20] gives

$$\frac{1}{i\hbar}[\mathbf{g}, \mathbf{f}] = \sum_{m=1}^{\infty} \frac{(-i\hbar)^{m-1}}{m!} \left[\frac{\partial^m \mathbf{g}}{\partial \mathbf{Q}^m} \frac{\partial^m \mathbf{f}}{\partial \mathbf{P}^m} - \frac{\partial^m \mathbf{f}}{\partial \mathbf{Q}^m} \frac{\partial^m \mathbf{g}}{\partial \mathbf{P}^m} \right]$$

of the particular **g**, **f** in the matrix map **T** of (2.3), since the classical limit of Eq. (2.6) is Eq. (2.2), according to Eq. (2.5a). Note that whenever **g** (or **f**) is linear, the AP property (2.6) is *the exact and unique* qm extension of the classical ap property (2.2), for any value of $\hbar$ (see footnote 2). No *limit* (2.5a) is needed in that case.

It follows from Eq. (2.6) that the *qm commutator (2.4) is a mapping constant of the motion:*

$$[\mathbf{Q}_{t+1}, \mathbf{P}_{t+1}] = [\mathbf{g}, \mathbf{f}] = i\hbar\mathbf{I} = [\mathbf{Q}_t, \mathbf{P}_t] \tag{2.7}$$

under our AP matrix map **T** (2.3), (2.6), i.e., the qm commutator (2.4) holds for all (integer) values of t.

A second essential "axiom" of quantum mechanics is that there exists a linear operator $\mathbf{U}_t$ governing the evolution over a time t, e.g., $\mathbf{Q}_t = \mathbf{U}_t\mathbf{Q}_0\mathbf{U}_{-t}$ (hence $\mathbf{U}_{-t}\mathbf{U}_t = \mathbf{I}$), which is characteristic of the system[17] (autonomous here), i.e., *one* and the same operator $\mathbf{U}_t$ for all *all* qm observables. It is from this "axiom" that the Schrödinger equation and Heisenberg equation are usually inferred (see Chapter 5, §27, 28 of ref. 17). The $\mathbf{U}_t$ does depend on the initial conditions, $\mathbf{U}_t = \mathbf{U}_t(\mathbf{Q}_0\mathbf{P}_0)$. For the time evolution of an arbitrary observable $\mathbf{Y}_t$ this means

$$\mathbf{Y}_t = \mathbf{U}_t\mathbf{Y}_0\mathbf{U}_{-t} \tag{2.8a}$$

whence

$$\mathbf{U}_t = (\mathbf{U}_1)^t, \qquad (\mathbf{U}_1^\dagger\mathbf{U}_1)\,\mathbf{Y}_0 = \mathbf{Y}_0(\mathbf{U}_1^\dagger\mathbf{U}_1) \tag{2.8b}$$

since $\mathbf{Y}_t^\dagger = \mathbf{Y}_t$, So $(\mathbf{U}_1^\dagger\mathbf{U}_1)$ commutes with all observables, and therefore with all (complex) matrices.[17,23] Hence $(\mathbf{U}_1^\dagger\mathbf{U}_1)$ must be proportional to the identity **I**. Since the proportionality constant has no effect on (2.8a), we take $\mathbf{U}_1^\dagger\mathbf{U}_1 = \mathbf{I}$, i.e., $\mathbf{U}_1$ is a "unitary" matrix. A unitary matrix can be rewritten as

$$\mathbf{U}_1 \equiv e^{i\mathbf{v}}, \qquad \text{with} \quad \mathbf{v}^\dagger = \mathbf{v} \tag{2.8c}$$

the "polar decomposition" of $\mathbf{U}_1$.[23] For our map variables this produces the representation

$$\mathbf{Q}_t = e^{i\mathbf{v}t}\mathbf{Q}e^{-i\mathbf{v}t}, \qquad \mathbf{P}_t = e^{i\mathbf{v}t}\mathbf{P}e^{-i\mathbf{v}t}, \qquad t \equiv \ldots, -1, 0, 1, \ldots \tag{2.9}$$

where $\mathbf{Q} \equiv \mathbf{Q}_0$, $\mathbf{P} \equiv \mathbf{P}_0$. This representation of $\mathbf{Q}_t$, $\mathbf{P}_t$ is compatible with the time evolution under our matrix map and the qm commutator, as we see by substitution of Eq. (2.9) into Eqs. (2.3) and (2.4)–(2.7) and multiplying out $e^{\pm i\mathbf{v}t}$ terms.

If our map (2.1) is the surface of section map for a "larger" Hamiltonian system[11,12] with Hamiltonian H and qm extension $\boldsymbol{H}$, observables can be similarly expressed,[13–15,17,18] but now in real time τ, as

$$\mathbf{Y}(\tau) \equiv e^{i\boldsymbol{H}\tau/\hbar}\mathbf{Y}e^{-i\boldsymbol{H}\tau/\hbar} \tag{2.10}$$

The $\boldsymbol{\nu}$ in the (mapping) representation (2.8), (2.9) can thus be interpreted as $\mathbf{H}/\hbar$, the part of some "larger" Hamiltonian $\boldsymbol{H}/\hbar$, which generates our map (2.1), (2.3) as its surface of section map. From Eq. (2.10) one finds the familiar Heisenberg equations:

$$d\mathbf{Y}(\tau)/d\tau = (1/i\hbar)[\mathbf{Y}(\tau), \boldsymbol{H}] \tag{2.11}$$

whence

$$\begin{aligned} d\mathbf{Q}(\tau)/d\tau &= (1/i\hbar)\{\boldsymbol{Q}(\tau), \boldsymbol{H}\} = \partial\boldsymbol{H}/\partial\mathbf{P} \\ d\mathbf{P}(\tau)/d\tau &= (1/i\hbar)\{\mathbf{P}(\tau), \boldsymbol{H}\} = -\partial\boldsymbol{H}/\partial\mathbf{Q} \end{aligned} \tag{2.12}$$

[cf. Eq. (2.5b)], i.e., the canonical, e.g., *nonlinear*, equations of motion for $\mathbf{Q}(\tau)$, $\mathbf{P}(\tau)$ under the "larger" Hamiltonian $\boldsymbol{H}/\hbar$.

The $\boldsymbol{\nu}$ representation (2.9) immediately yields the corresponding equations for our quantum map $\boldsymbol{T}$ in (2.3), (2.6):

$$\mathbf{Y}_{t+1} - \mathbf{Y}_t = -[\mathbf{Y}_t, e^{i\boldsymbol{\nu}}]\, e^{-i\boldsymbol{\nu}}, \qquad \mathbf{Y}_t = e^{i\boldsymbol{\nu}t}\mathbf{Y}e^{-i\boldsymbol{\nu}t} \tag{2.13}$$

whence

$$\mathbf{g}(\mathbf{Q}_t, \mathbf{P}_t) - \mathbf{Q}_t = \mathbf{Q}_{t+1} - \mathbf{Q}_t = -i\hbar(\partial e^{i\boldsymbol{\nu}}/\partial\mathbf{P}_t)\, e^{-i\boldsymbol{\nu}} \tag{2.14a}$$

$$\mathbf{f}(\mathbf{Q}_t, \mathbf{P}_t) - \mathbf{P}_t = \mathbf{P}_{t+1} - \mathbf{P}_t = i\hbar(\partial e^{i\boldsymbol{\nu}}/\partial\mathbf{Q}_t)\, e^{-i\boldsymbol{\nu}} \tag{2.14b}$$

[cf. Eq. (2.5b)]. Note however that this $\mathbf{U} = e^{i\boldsymbol{\nu}(\mathbf{Q}_t, \mathbf{P}_t)}$ usually is not differentiable and certainly not analytic, for a nonintegrable map.[8,10] The same is true for its "full" Hamiltonian H, if we (try to) reexpress H as a function of q_t, p_t only.[8,10] A differentiable, but nonanalytic, representation of $\mathbf{U}$ does exist for certain maps.[27,28]

Since we are interested in the energy levels, i.e., the eigenvalues of $\mathbf{H} \equiv \hbar\boldsymbol{\nu}$, we take it that we are already working in a "representation" of all variables in which $\boldsymbol{\nu}$ and $\mathbf{H}$ are diagonal:

$$\langle k|\, \boldsymbol{\nu}\, |n\rangle \equiv (\boldsymbol{\nu})_{k,n} \equiv \delta_{k,n}\nu_n \equiv \delta_{k,n}E_n/\hbar \equiv \langle k|\, \mathbf{H}\, |n\rangle/\hbar, \qquad k, n = 0, 1, \ldots \tag{2.15}$$

with eigenstates $|n\rangle$, the usual scalar product and real ν_n, E_n.[17]

According to Eq. (2.13), $\mathbf{Y}_t$ is a mapping constant of the motion $\mathbf{Y}_{t+1} = \mathbf{Y}_t$ *if and only if* $\mathbf{Y}_t$ *commutes with* $e^{i\boldsymbol{\nu}}$. Since $\boldsymbol{\nu}$ is diagonal, so is $e^{i\boldsymbol{\nu}}$, and

$$[e^{i\boldsymbol{\nu}}, \mathbf{Y}]_{k,n} = Y_{k,n}(e^{i\nu_k} - e^{i\nu_n})$$

Hence, in general, this commutator vanishes if and only if $\mathbf{Y}$ is diagonal as well. If $e^{i\mathbf{v}}$ is degenerate, i.e., if $v_k = v_n + p2\pi$ for some integers k, n, p, the $\mathbf{Y}_t$ clearly need not be diagonal in order to be a mapping constant of the motion. Thus, we find that a variable is a mapping constant of the motion if and only if the variable is diagonal ("only if": as long as $e^{i\mathbf{v}}$ is not degenerate). Note that according to (2.11), $\mathbf{Y}(\tau)$ is a constant of the motion for *all* differential equations of motion generated by the "larger" Hamiltonian $\boldsymbol{H}$ if and only if $\mathbf{Y}$ is diagonal ("only if": as long as $\boldsymbol{H}/\hbar$ is not degenerate).

Translating an argument of Jordan[13] to our mapping case, we apply the above to our qm commutators. Equation (2.7) states that this commutator is a (mapping) constant of the motion. Hence, in general, the qm commutator is diagonal and we need to satisfy *only* the *diagonal* elements of the commutator (2.4):

$$\operatorname{Im}\left(\sum_{m=0}^{\infty} Q_{n,m}P_{m,n}\right) = \tfrac{1}{2}\hbar, \qquad n, m = 0, 1,... \tag{2.16}$$

where Im denotes the imaginary part, $P_{n,m} \equiv (\mathbf{P})_{n,m}$, etc. For our considerations all variables are without physical units and $\hbar$ merely is some number, which we choose small.

Substituting the $\mathbf{v}$ representation (2.9) into our matrix map $\mathbf{T}$ in (2.3), we find the time-*independent* matrix "map"

$$e^{i\mathbf{v}}\mathbf{Q}e^{-i\mathbf{v}} = \mathbf{g}(\mathbf{Q}, \mathbf{P}), \qquad e^{i\mathbf{v}}\mathbf{P}e^{-i\mathbf{v}} = \mathbf{f}(\mathbf{Q}, \mathbf{P}) \tag{2.17a}$$

$$\begin{aligned} Q_{k,n}e^{i(v_k - v_n)} &= g_{k,n}(\mathbf{Q}, \mathbf{P}) \\ P_{k,n}e^{i(v_k - v_n)} &= f_{k,n}(\mathbf{Q}, \mathbf{P}) \end{aligned} \tag{2.17b}$$

$(k, n = 0, 1,...)$ after multiplying out $e^{\pm i\mathbf{v}t}$ terms, where $g_{k,n} \equiv (\mathbf{g})_{k,n}$, etc. Hence, we need to solve simultaneously Eqs. (2.16), (2.17b) for $\mathbf{Q}, \mathbf{P}, \mathbf{v}$. Simple ways of doing this for polynomial maps will be given in the next sections.

3. POLYNOMIAL AND LINEAR MAPS

The two first-order difference equations of the map T in (2.1) can be combined into one second-order difference equation with only *one* variable x in many cases when T is polynomial (refs. 24, 25; also ref. 9, Appendix A, pp. 114–117). From now on I prefer to work with ap maps in this standard form, since only one matrix $\mathbf{X}$ needs to be solved, besides $\mathbf{v}$. Their quantization rules are discussed first. In Section 3.2, I quantize the linear "harmonic" ("elliptic") map exactly, using the mapping quantization introduced

in Sections 2 and 3.1. In Section 3.3 these results are compared with those obtained from the usual quantization of the "larger" (4-dimensional) Hamiltonian for a pair of harmonic oscillators whose surface of section map reproduces our linear map exactly. The results are the "same": Of course our (2-dimensional) mapping quantization does not yield levels due to motion "*perpendicular*" to the plane of the map. Yet, our mapping quantization explicitly allows for the existence of such perpendicular levels and eigenstates, uncoupled to those of the map (i.e., with vanishing transition elements between them). If desired, perpendicular states and levels may be added afterward. The exact, and familiar, results here provide the simplest possible example of our mapping quantization.

3.1. Quantizing Polynomial Maps; Second-Order Difference Equations

If both f and g are polynomials whose orders are relative prime, the two first-order difference equations (2.1) can in general be reduced to one *second*-order difference equation[9,24] in one variable via a transformation from q_t, p_t variables to new variables x_t, y_t, with the particular property[9,24]

$$y_{t+1} = ax_t + bx_{t+1}, \qquad a \neq 0, \quad t = \ldots, -1, 0, 1, \ldots \tag{3.1}$$

where a and b are (constant) functions of the parameters in f and g. It is this relation that allows us to replace all y variables with x variables everywhere and end up with a second-order difference equation in just one variable. Such variables can always be found for all linear, all quadratic, and many polynomial ap maps.[9,24] For all these maps the AP property (2.6) is the unique qm extension of the classical area-preserving property (2.2), since one of the (2) mapping equations is linear. Here we shall employ either the cubic map[8–10]

$$x_{t+1} + x_{t-1} = 2Cx_t + 2x_t^3, \qquad t = \ldots, -1, 0, 1, \ldots \tag{3.2}$$

or its linear part. Equation (3.2) is an area-preserving mapping[8–12] from the x_{t-1}, x_t plane to the x_t, x_{t+1} plane. Hence, the axes of our phase plane are x_t and x_{t+1} for all integer t values. Besides the **v** representation of x_t

$$\mathbf{X}_t \equiv e^{ivt}\mathbf{X}e^{-ivt}, \qquad t = \ldots, -1, 0, 1, \ldots \tag{3.3}$$

with $\mathbf{X}_t^\dagger = \mathbf{X}_t$ [cf. Eq. (2.9)] and the matrix representation of the map (3.2), i.e., Eqs. (3.10), (4.2), we need the qm commutator of $\mathbf{X}_{t-1}$ with $\mathbf{X}_t$. From the linear relation (3.1) we find

$$[\mathbf{Y}_t, \mathbf{X}_t] = a[\mathbf{X}_{t-1}, \mathbf{X}_t], \qquad t = \ldots, -1, 0, 1, \ldots \tag{3.4}$$

Since x_t and y_t were independent functions of q_t and p_t [e.q., Eqs. (3.21), (3.17)],[9,24,25] the left-hand side of Eq. (3.4) is some constant multiple d ($\neq 0$, in general) of $[\mathbf{Q}_t, \mathbf{P}_t]$ in (2.4), e.g., Eq. (3.22). Thus, we obtain a qm commutator:

$$[\mathbf{X}_{t-1}, \mathbf{X}_t] = (d/a)(i\hbar\mathbf{I}) \equiv i\underline{\hbar}\mathbf{I}, \qquad t = \ldots, -1, 0, 1, \ldots \tag{3.5a}$$

where, for convenience, I use the symbol $\underline{\hbar}$ for Planck's constant h times $d/(2a\pi)$. We shall only use

$$[\mathbf{X}_0, \mathbf{X}_1] = [\mathbf{X}, e^{i\nu}\mathbf{X}e^{-i\nu}] = i\underline{\hbar}\mathbf{I} \tag{3.5b}$$

[cf. (3.3)] after multiplying out $e^{\pm i\nu t}$. Since we pay no attention to physical units in this paper there is no practical difference here between $\hbar$ and $\underline{\hbar}$. Note, however, that *our $\underline{\hbar}$ depends on the parameters* of the mapping (2.3), or the Hamiltonian, via a and d, e.g., Eq. (3.21). For the linear, "harmonic" map the relation between $\underline{\hbar}$ and $\hbar$ will be derived in Eq. (3.22b). Substitution of one of our matrix maps, (3.10) or (4.2), easily shows that $[\mathbf{X}_{t-1}, \mathbf{X}_t] = [\mathbf{X}_t, \mathbf{X}_{t+1}]$, etc. So we see again that the qm commutator is a mapping constant of the motion. We saw before, between Eqs. (2.15) and (2.16), that this results in the off-diagonal elements of the qm commutator (2.4), (3.5) vanishing in general. Hence, in general, we need to satisfy *only* the *diagonal* elements of Eq. (3.5b):

$$\sum_{m=0}^{\infty} |X_{n,m}|^2 \sin(\nu_m - \nu_n) = \tfrac{1}{2}\underline{\hbar} \qquad n = 0, 1, \ldots \tag{3.5c}$$

[cf. Eq. (2.16)]. As our $\underline{\hbar}$ has already absorbed some parameters, this version of the Thomas–Reiche–Kuhn sum rule[13-15] *for our transformed map variables* looks unfamiliar: Normally, there is no sine function, but just $\nu_m - \nu_n$. Yet we find in Section 3.2 that the exact, familiar results are obtained from Eq. (3.5c), using Eq. (3.22b). Similarly, some other results in Section 3.2 may look unfamiliar.

3.2. Quantum Levels of the Linear Map

Consider an area-preserving, linear "elliptic" mapping:

$$x_{t+1} = L_{11}x_t + L_{12}y_t, \qquad y_{t+1} = L_{21}x_t + L_{22}y_t \tag{3.6a}$$

with $t = \ldots, -1, 0, 1, \ldots$, and

$$\det(\mathbf{L}) = 1 \qquad \text{and} \qquad |\text{trace}(\mathbf{L})| < 2 \tag{3.6b}$$

[cf. Eq. (2.2)], where $L_{11} = (\mathbf{L})_{1,1}$, etc. Eliminate the y's from these equations and combine them into one second-order equation[8–10,24,25]

$$x_{t+1} + x_{t-1} = 2Cx_t \qquad \text{with} \quad 2C \equiv \text{trace}(\mathbf{L}), \quad t = \ldots, -1, 0, 1, \ldots \tag{3.7}$$

where I lowered all t by 1. One is interested in the stable, "elliptic," case, where $|C| < 1$. The exact classical solution of the "harmonic" map (3.7) then is

$$x_t = ze^{i\alpha t} + \bar{z}e^{-i\alpha t} \equiv r \sin(\alpha t + \phi), \qquad \text{with } \cos\alpha \equiv C, \quad 0 < \alpha < \pi \tag{3.8}$$

real α, r, ϕ, complex z, and integer t cf. Eq. (4.5). Since a linear mapping is "Integrable,"[8-10] it has a smooth, real, analytic constant of the motion, a "*second*" integral J:

$$J \equiv \tfrac{1}{2}(x_t^2 + x_{t+1}^2) - Cx_t x_{t+1} = \text{constant}_t, \qquad t = \ldots, -1, 0, 1, \ldots \tag{3.9}$$

in addition to the full (4-dimensional) Hamiltonian H in (3.19a), from which our surface of section map (3.7) can be derived.[11,12]

The matrix representation of our map (3.7) is

$$\mathbf{X}_{t+1} + \mathbf{X}_{t-1} = 2C\mathbf{X}_t, \qquad t = \ldots, -1, 0, 1, \ldots \tag{3.10}$$

Substituting the $\mathbf{v}$ representation (3.3), we obtain the time-*independent* matrix "map":

$$e^{i\mathbf{v}}\mathbf{X}e^{-i\mathbf{v}} + e^{-i\mathbf{v}}\mathbf{X}e^{i\mathbf{v}} = 2C\mathbf{X} \tag{3.11a}$$

i.e.,

$$X_{k,n}\cos(v_k - v_n) = \cos(\alpha)\, X_{k,n}, \qquad k, n = 0, 1, \ldots \tag{3.11b}$$

after multiplying out $e^{\pm iv_t}$ terms [cf. Eq. (2.17)]. The qm commutator (3.5c) and Eq. (3.11b) are solved by

$$v_n = (n + \tfrac{1}{2})\,\alpha, \quad X_{n+1,n} = (n+1)^{1/2}\,(\underline{\hbar}/2 \sin\alpha)^{1/2}\, e^{i\phi_n} = \bar{X}_{n,n+1}, \quad n = 0, 1, \ldots \tag{3.12}$$

where ϕ_n is an arbitrary phase angle and the $\{X_{n+1,n} = \bar{X}_{n,n+1}\}$ are the only nonvanishing elements of $\mathbf{X}$. The level $v_0 = \frac{1}{2}\alpha$ is chosen somewhat arbitrarily. Formally, the solution (3.12) is not unique since *vanishing* (!) off-diagonals might be inserted between the first off-diagonal (3.12) and the main diagonal. More generally, a "direct product"[22] or "direct sum"[22] of the $\mathbf{X}$ with any identity matrix is a solution as well. This amounts to inserting new levels, *uncoupled* to the ones above, and allows for any "perpendicular" degrees of freedom, as mentioned earlier. As our $\underline{\hbar}$ has already

absorbed some parameters, this version (3.12) of the matrix elements,[13–15] *for our transformed map variables*, looks unfamiliar. Yet, using Eq. (3.22b), we recover the familiar results. The solution (3.12) also diagonalizes the second Integral J of (3.9) [we symmetrized its last term into $\frac{1}{2}C(x_t x_{t+1} + x_{t+1} x_t)$]:

$$J_{k,n} = \delta_{k,n}(n + \tfrac{1}{2})\, \underline{\hbar} \sin \alpha, \qquad n = 0, 1, \ldots \tag{3.13}$$

Much of this appears similar to the levels of the familiar harmonic oscillator.[15,17] Note that the diagonal second integral **J** above [even after reexpressing $\underline{\hbar}$ in terms of $\hbar$ via Eq. (3.22b)] is different from the diagonal "mapping Hamiltonian" whose levels we find, using Eqs. (3.12), (2.15),

$$\mathrm{H}_{k,n} = \delta_{k,n} E_n \equiv \delta_{k,n} \hbar \nu_n = \delta_{k,n}(n + \tfrac{1}{2})\, \hbar\alpha, \qquad n = 0, 1, \ldots \tag{3.14}$$

These are harmonic oscillator levels indeed [reexpressed in terms of our "rotation number" α (3.8), a *ratio* of two frequencies, times 2π; cf. (3.17b)][13–15] The "zero-point" levels $\mathrm{H}_{0,0}$ and $J_{0,0}$ are chosen somewhat arbitrarily, since constants can be added to H and/or J without any effect on the equations of motion. For our harmonic map the **J** of (3.13) and **H** of (3.14) (whence $e^{i\nu}$) obviously are commuting operators. In this harmonic case the $e^{i\nu}$ is degenerate if and only if $\alpha/2\pi$ is a rational number. Note that in the case of degeneracy the $\{e^{i\nu_n t}\}$ contain only a finite number of independent exponentials [as do the $\{e^{i\varepsilon_n \tau}\}$ in (3.20)] and *mapping* constants of the motion need not be diagonal. The mixed state solutions, where $\boldsymbol{\nu}$ need not be diagonal are given in Eq. (4.5).

3.3. Comparison with the Familiar Quantization of the Full Hamiltonian

Consider two harmonic oscillators

$$d^2x(\tau)/d\tau^2 = -\omega_x^2 x(\tau), \qquad d^2y(\tau)/d\tau^2 = -\omega_y^2 y(\tau) \tag{3.15}$$

describing a harmonic oscillator of mass m in two spatial dimensions and real time τ. The usual classical solution is of course

$$x(\tau) = A \sin(\omega_x \tau + \xi), \qquad y(\tau) = B \sin(\omega_y \tau + \eta) \tag{3.16}$$

Simple trigonometry shows that

$$x(\tau + 2\pi/\omega_y) + x(\tau - 2\pi/\omega_y) = 2Cx(\tau) \tag{3.17a}$$

with

$$C = \cos(\alpha), \qquad \alpha/2\pi \equiv \omega_x/\omega_y \tag{3.17b}$$

Hence Eq. (3.17a) is the same as our linear map (3.7) once we change the time unit of τ at discrete moments in time according to

$$t \equiv \tau(\omega_y/2\pi) \qquad \text{and} \qquad x_t \equiv x(\tau) \qquad \text{at} \quad t = \ldots, -1, 0, 1, \ldots \tag{3.18}$$

The Hamiltonian H of the 2-dimensional harmonic oscillator (3.15) is

$$H \equiv (p_x^2 + p_y^2)/2m + \tfrac{1}{2}m(\omega_x^2 x^2 + \omega_y^2 y^2) \tag{3.19a}$$

with $p_x = m\, dx/d\tau$, etc. Its familiar quantum levels are[13-15]

$$\varepsilon_{n_x, n_y} = {}^x\varepsilon_{n_x} + {}^y\varepsilon_{n_y}, \qquad \text{with} \quad {}^x\varepsilon_n \equiv (n + \tfrac{1}{2})\,\hbar\omega_x, \qquad n, n_x, n_y = 0, 1, \ldots \tag{3.19b}$$

and ${}^y\varepsilon_n$ is defined analogously. Comparison with the levels E_n [Eq. (3.14)] of our mapping shows

$${}^x\varepsilon_n = (\omega_y/2\pi)\, E_n$$

whence

$$\exp(i\, {}^x\varepsilon_n \tau) = \exp(iE_n t) \qquad \text{at} \quad t = \ldots, -1, 0, 1, \ldots \tag{3.20}$$

[cf. (3.18), (3.17b)]. Hence, our mapping quantization yields the exact energy levels due to the x motion (apart from a trivial change of time/energy scale depending on the perpendicular motion).

Finally, we express the $\underline{\hbar}$ we used for the harmonic map in (3.12), (3, 13), and (3.5) in terms of $\hbar$. Note from Eqs. (3.16)–(3.18) that

$$x_{t-1} \equiv x(\tau - 2\pi/\omega_y) = Cx(\tau) - p_x(\tau)\sin(\alpha)/m\omega_x \tag{3.21}$$

Thus the qm commutator (3.5a) for $\mathbf{X}$ becomes

$$[\mathbf{X}_{t-1}, \mathbf{X}_t] = [\mathbf{X}, \mathbf{P}_x]\sin(\alpha)/m\omega_x = i\underline{\hbar}\mathbf{I} \tag{3.22a}$$

whence

$$\underline{\hbar} = \hbar \sin(\alpha)/m\omega_x \tag{3.22b}$$

for our harmonic map (3.10). Substitution of this $\underline{\hbar}$ produces the more familiar forms of the matrix elements $x_{n+1,n}$ [Eq. (3.12)] and of the Thomas–Reiche–Kuhn rules (3.5c).[13-15]

In the next section I treat a cubic "nonintegrable" mapping, cf. Section 3.1 of Moser's 1968 article reprinted in ref. 10.[8-10] There is, by definition, no second integral in addition to the Hamiltonian. Hence, the analogue of $\mathbf{H}$ survives but the analogue of $\mathbf{J}$ disappears.

4. QUANTUM LEVELS OF THE CUBIC MAP

Having introduced the procedure and notation, I proceed with the quantization of the area-preserving cubic map[8,9,10]

$$x_{t+1} + x_{t-1} = 2Cx_t + 2x_t^3, \qquad t = \ldots, -1, 0, 1, \ldots \tag{4.1}$$

Note that Eq. (4.1) has *unbounded* solutions as well as bounded ones (e.g., at $C=0$, take $x_0=10=x_1$). Hence, in the quantum version, one expects there to be "tunneling" and no truly stationary ($t\to\infty$) solutions at all![15,17,18] Thus, technically, there are no truly discrete levels with real v_n (2.15). In practice however, I truncate the $\mathbf{X}$, $\mathbf{H}$, and $\mathbf{v}$ to finite $(N+1)\times(N+1)$ matrices. As long as the truncation is such that for all our states $|n\rangle$ the rms displacements $\langle n|\,\mathbf{X}^2\,|n\rangle^{1/2}$ (with $n\leqslant N$) stay well within the classically "bounded" region, one expects *not* to see the effects of tunneling in practice, since there is then no coupling to higher energies (with classically unbounded displacements). This is the analogue of the usual WKB approximation in the Schrödinger picture, or perturbation theory over only a finite number of "levels." It is usually "justified" by the (generally) extremely long times before tunneling takes place, i.e., experimentally one would see "levels" indeed.[13-15] The truncation amounts to a "cutoff" of the map (4.1) and the $\mathbf{H}$. One of the preliminary results below is that one gets numerical convergence up to states with rms displacements which classically are reached at (initial displacements of) about 30% of the distance to the edge of the classically bounded region (and about 55% for the quadratic map). This is reasonable at this preliminary stage, since one would expect an "eigenfunction" (in a Schrödinger approach) to already have an appreciable value in the *unbounded* region, once its peak gets that close to the edge of the bounded region.

The matrix representation of the cubic map (4.1) is

$$\mathbf{X}_{t+1}+\mathbf{X}_{t-1}=2C\mathbf{X}_t+2\mathbf{X}_t^3, \qquad t=\ldots,-1,0,1,\ldots \tag{4.2}$$

[cf. (2.3), (3.10)]. Using the $\mathbf{v}$ representation and multiplying out the $e^{\pm i\mathbf{v}t}$ terms, we obtain the time-*independent* matrix "map"

$$e^{i\mathbf{v}}\mathbf{X}e^{-i\mathbf{v}}+e^{-i\mathbf{v}}\mathbf{X}e^{i\mathbf{v}}=2C\mathbf{X}+2\mathbf{X}^3 \tag{4.3a}$$

i.e.,

$$X_{k,n}\cos(v_k-v_n)=\cos(\alpha)\,X_{k,n}+\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}X_{k,m}X_{m,j}X_{j,n}, \qquad k,m,j,n=0,1,\ldots \tag{4.3b}$$

analogous to the procedure used for the linear map (3.11).

4.1. Iterative Solutions

Truncation of the matrices creates some practical problems. Now the elements of the qm commutator (3.5b), (2.4), (2.10) *cannot all* be satisfied exactly,[13] e.g., note that for *finite* matrices *trace*$\{[\mathbf{X}_0,\mathbf{X}_1]\}$ must vanish, whereas the rhs of Eqs. (3.5b), (2.4) requires it to be *nonvanishing*. So, in our $(N+1)\times(N+1)$ truncation the trace does vanish, i.e., the last diagonal element (at N,N) of the qm commutator cannot be satisfied and

is not utilized. Several other properties of finite matrices do not necessarily hold for *infinite* matrices.[13–15, 19] The truncated versions of Eqs. (4.3b) and (3.5c) present $(N+1)^2+N$ equations for the $(N+1)^2+N$ (real) variables in the truncated $\mathbf{X}$, $\mathbf{v}$ (v_0 is arbitrary).

We iterate the equations as follows. Starting with some initial estimate $\mathbf{X}^{(0)}$, usually the harmonic one (3.12), we evaluate the rhs of the matrix map (4.3b) for all k and n values ($N \geqslant k \geqslant n \geqslant 0$: all matrices are Hermitian, $k-n$ is odd). Second, we extract from the lhs of (4.3b) at $k=n+1$ all level separations:

$$\Delta v_{n+1} \equiv v_{n+1} - v_n \qquad (\equiv \Delta E_{n+1}/\hbar), \qquad n=0,\ldots,N-1 \tag{4.4}$$

using the old $X^{(0)}_{n+1,n}$ (iterations were considered "divergent" when $|\cos(\Delta v_n)| \geqslant 1$; or if e^{iv} had been degenerate). Third, employing these level separations, we obtain from Eq. (4.3b) new estimates $X^{(1)}_{k,n}$ at all other k and n values ($n=0,\ldots,N$; $k=n+1, n+3,\ldots,N$), occasionally employing some of the techniques of ref. 26 to accelerate the convergence of the iterations, e.g., carrying all linear terms of (4.3b) over to its lhs. Fourth, using all the new estimates so far, we evaluate the n,n diagonal element of the qm commutator (3.5c) and solve it for the as yet unknown $|X^{(1)}_{n,n+1}|^2$ at $n=0,\ldots,N-1$. The resulting phase angle ϕ_n [cf. Eq. (3.12)] is set equal to zero in all our calculations. Having thus come "full circle" in our iteration, we evaluate the off-diagonal elements of the commutator (3.5b) to check how small they are and how well our finite truncation approximates the infinite commutator matrix. The worst ones are near the Nth row/column, the best ones near the lowest rows/columns. We iterate until *each* of these off-diagonal elements is absolutely smaller than $10^{-7}\underline{\hbar}$ (usually 40–70 iterations). By that time the relative changes in the matrix elements $X_{k,n}$ and in the v_n are of order 10^{-15}. Some preliminary results are listed in Table I.

Due to boundary effects of the truncation at $n=N$ [e.g., the N, N element of the commutator (3.5) is not utilized], the highest levels with reasonably consistent results (over different values of $\underline{\hbar}$) appear to be $n=N-2$ (or $N-3$), whence the use of $N-2$ in Table I. Since the $\underline{\hbar}$ of (3.5a) has absorbed some parameters (with unknown values), I used the maximum $\underline{\hbar}$ value which still appeared to yield numerical convergence. While this introduces some "noise" in the value of $\underline{\hbar}$ used, note that the product $(N-2)\underline{\hbar}$ and the highest level $\underline{\hbar}(v_{N-2}-v_0)/2\pi$ remain relatively constant under large changes in N (cf. the second and fourth rows of Table I), which is as it should be (since $v_n = E_n/\hbar \propto E_n/\underline{\hbar}$). Hence, extremely large values of N are not required in order to get a good impression of the qm properties of the map! Comparing the fourth and fifth rows of Table I, where $v_{N-2}-v_0$ is given for the cubic map and for the

Table I. Quantization of the Cubic Map (4.1)–(4.3) at Maximum $\underline{\hbar}$ Values (3.5) for a Given Truncation Level N[a]

	$N=202$	$N=69$	$N=36$	$N=9$	Comment
$\sim\underline{\hbar}_{\max}$ (rel. error $\approx 0.2\%$)	4.895×10^{-5}	1.5664×10^{-4}	3.289×10^{-4}	1.0798×10^{-2}	Maximum $\underline{\hbar}$ for numerical convergence
$\sim(N-2)\,\underline{\hbar}$	8.779×10^{-3}	1.0495×10^{-2}	1.1185×10^{-2}	7.559×10^{-2}	Compare 3 rows below
$\sim\Delta\nu_{N-2}/2\pi$	0.401805	0.400647	0.40019	0.36868	$\alpha/2\pi=0.408298$, Eqs. (3.8), (3.12)
$\sim\underline{\hbar}(\nu_{N-2}-\nu_0)/2\pi$ "cubic" map (4.2)	3.555×10^{-3}	4.243×10^{-3}	4.5185×10^{-3}	2.898×10^{-2}	$\nu_n=E_n/\hbar$
$\sim\underline{\hbar}(\nu_{N-2}-\nu_0)/2\pi$ linear map (3.10)	3.997×10^{-3}	4.285×10^{-3}	4.5658×10^{-3}	3.086×10^{-2}	$=(N-2)\,\underline{\hbar}\alpha/2\pi$ [cf. (3.8)]
$\langle N-2\vert\,\mathbf{X}^2\,\vert N-2\rangle^{1/2}$	0.1233089	0.1344395[b]	0.13897	0.32924	Classically maximum rms: 0.40452[b]

[a] The value of $\underline{\hbar}$ used everywhere is the $\underline{\hbar}_{\max}$ (see text) of the first row. All levels ν_n and other results are for $C=-0.83855$ [i.e., $\alpha/2\pi=0.408298...$; cf. Eqs. (3.8), (3.12)]. Equations (4.3b) and (3.5c) are iterated until each off-diagonal element of the qm commutator is absolutely smaller than $10^{-7}\underline{\hbar}$.

[b] Classically, the rms displacement is 0.1341 (and the winding number $\rho\approx0.40$; cf. $\Delta\nu_{N-2}/2\pi$ above) at $x_0=x_1\approx0.057$, i.e., at $\approx24\%$ of the distance to the edge of the bounded region ($\rho\approx0.34375$ there).

harmonic map, we see that the levels are lower than, but still close to, the levels of the harmonic map, i.e., $\nu_n - \nu_0 \lesssim n\alpha$ [Eq. (3.12)], where $\alpha/2\pi$ is the classical winding number (3.8) at the origin of the phase plane. At $n = N-2$ this difference is "larger" than at lower n values, but is still small. We find that the level splitting $\Delta\nu_{n+1} (\equiv \nu_{n+1} - \nu_n)$ decreases only slightly from the harmonic value, $\alpha = 0.408298(2\pi)$, at low n values, to $\approx 0.401(2\pi)$ at $n = N-2$ (see row 3 of Table I). It is easily checked that the *classical* winding numbers ρ also decrease only slightly, from $\alpha/2\pi$ at the origin of the phase plane to $\rho \approx 0.40$ at $x_0 = x_1 \approx 0.057$ (i.e., at 24% of the distance to the edge of the classically bounded region; $\rho \approx 0.34375...$ there). It is for these initial values that the classical rms displacement, $[= \lim_{q\to\infty} (\sum_{t=0}^{q} x_t^2 q)^{1/2}]$ equals the largest quantum rms displacement ($\langle N-2| \mathbf{X}^2 |N-2\rangle^{1/2}$) (see row 6 of Table I), whence our comparison of classical and qm results at *these* initial conditions. It is interesting that some qm results are close to the corresponding classical results for the cubic map.

4.2. Time Evolution of Mixed States

Here I consider a general "mixed state" representation, in which the $\mathbf{v}$ (2.15) need not be diagonal. Starting with any (Hermitian) choice of $\mathbf{X}_0$, $\mathbf{X}_1$ which satisfy the qm commutator (3.5), I simply iterate the cubic matrix map (4.2) to obtain $\mathbf{X}_2$, $\mathbf{X}_3$, and subsequent $\mathbf{X}_t$. These continue to satisfy the qm commutator, as shown earlier [cf. Eqs. (2.7), (3.5b), (3.5c)]. The exact mixed state solutions of the *linear* quantum map (3.10) are

$$\mathbf{X}_t = -\frac{\sin(\alpha(t-1))}{\sin(\alpha)}\mathbf{X}_0 + \frac{\sin(\alpha t)}{\sin(\alpha)}\mathbf{X}_1 \tag{4.5}$$

with any (Hermitian) $\mathbf{X}_0$, $\mathbf{X}_1$, satisfying the qm commutator (3.5). For the cubic quantum map (4.2) I numerically evaluate a particular mixed state. For $\mathbf{X}_0$ I chose the real-valued $\mathbf{X}_0$ (3.12) of the linear map at the same value of C. For $\mathbf{X}_1$ I modified the complex $\mathbf{X}_1$ (3.12), (3.3) of the linear map to satisfy $\mathbf{X}_{-1} = \mathbf{X}_1^T$ here as well [as would be the case for a real matrix $\mathbf{v}$ in (3.3)], whence $\mathbf{X}_{-t} = \mathbf{X}_t^T$ and

$$\mathrm{Sym}(\mathbf{X}_1) \equiv \tfrac{1}{2}(\mathbf{X}_1 + \mathbf{X}_1^T) \equiv C\mathbf{X}_0 + \mathbf{X}_0^3 \qquad \text{with the } \mathbf{X}_0 \text{ of Eq. (3.12)} \tag{4.6}$$

[cf. Eq. (4.2)], where Sym is the symmetric part of the matrix. The $\mathbf{X}_1$ of the linear map satisfies (4.6) without the cubic term. Adding $\mathbf{X}_0^3$ to the $\mathbf{X}_1$ of the linear map to provide $\mathbf{X}_1$ here does not change its qm commutator with $\mathbf{X}_0$.

Iterate the complex matrix map (4.2) out to $t = 2.5 \times 10^3$. Repeating this for several $\hbar$ values gives the maximum value $\hbar_{\max}$ for which the time-

Table II. Maximum $\underline{\hbar}$ Values to Keep X_t "Bounded" up to $t = 2.5 \times 10^3$ for a Given Truncation Level N^a

	$N = 103$	$N = 69$	$N = 36$	$N = 9$
$\sim \underline{\hbar}_{\max}$ (rel. error $\approx 0.2\%$)	1.1467×10^{-3}	1.733×10^{-3}	3.474×10^{-2}	1.622×10^{-2}
$(N-2)\,\underline{\hbar}_{\max}$	1.1581×10^{-1}	1.161×10^{-1}	1.181×10^{-1}	1.136×10^{-1}

[a] Compare these rows with the first three rows of Table I. All results are for $C = -0.83855$ and the initial conditions given in Eq. (4.6).

dependent solution $\mathbf{X}_t$ remains "bounded." A solution $\mathbf{X}_t$ is considered "unbounded" as soon as $\langle n | \mathbf{X}_t^2 | n \rangle > 2$. The choice of 2 is a safety factor: Classically the bounded x_t^2 values remain less than 0.42 and the (time average of the) "bounded" quantum rms values remains less than 0.4. The resulting $\underline{\hbar}_{\max}$ values are listed in Table II.

Note that this yields larger $\underline{\hbar}$ values than the ones in Table I, for which the preliminary ν-level scheme converged numerically. We might correspondingly increase the usable $\underline{\hbar}$ values in the ν scheme by implementing all of the techniques of ref. 26. This was not yet done here.

I intend to employ of the methods of Section 4.1 as well as Section 4.2 to study quantum chaos[4-7] and follow up other connections with classical nonlinear dynamics.

Some recent interesting and relevant articles are listed in refs. 27–32.

5. ACKNOWLEDGMENTS

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